Controllability of Diffusively Coupled Multi-Agent Systems

Ming Cao

University of Groningen
The Netherlands

Joint work with Shuo Zhang and Kanat Camlibel

Outline

- Diffusively coupled multi-agent system and its controllability
- Graph partitions
- Controllability of systems with general graphs and distance regular graphs
  - Single-leader case
  - Multi-leader case
- Leader selection

Controllability of multi-agent systems

• Assigning the roles of leaders and followers

Diffusively coupled multi-agent systems

The state of each agent is a scalar: x_i

Agents' dynamics are determined by diffusive couplings

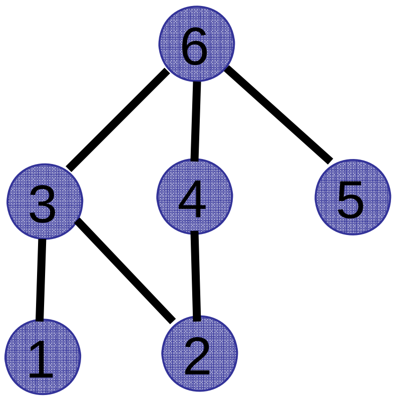
$$\dot{x}_i(t) = \sum_{j \in \mathcal{N}_i} (x_j(t) - x_i(t))$$

set of neighbors: time-invariant, symmetric

Overall dynamics

$$\dot{x}(t) = -Lx(t)$$

Fixed undirected neighbor relationship graphs $\mathbf{G} = (\mathcal{V}, \mathcal{E})$



$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 2 & -1 & -1 & 0 & 0 \\ -1 & -1 & 3 & 0 & 0 & -1 \\ 0 & -1 & 0 & 2 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & -1 & -1 & 3 \end{bmatrix}$$

$$D = \mathrm{diag}(1, 2, 3, 2, 1, 3)$$

Leaders vs. followers

Choose *m* leaders: $\mathcal{V}_l = \{l_1, l_2, \ldots, l_m\} \subseteq \mathcal{V}$

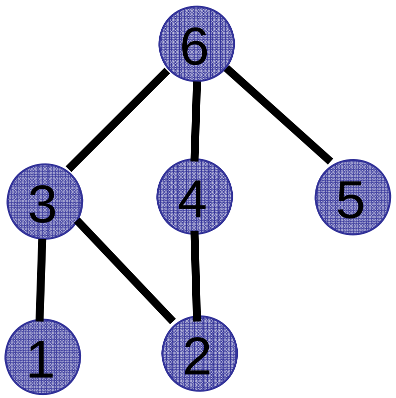
apply one control input at each leader $u = [u_1 \ldots u_m]^T$

Define $M = \mathbf{R}^{n\times m}$ where $M_{ij} = \begin{cases} 1 & \text{if } i = l_j \\ 0 & \text{otherwise} \end{cases}$

Overall dynamics

$$\dot{x}(t) = -Lx(t) + Mu$$

Example



Leaders vs. followers

Choose *m* leaders: $\mathcal{V}_l = \{l_1, l_2, \ldots, l_m\} \subseteq \mathcal{V}$

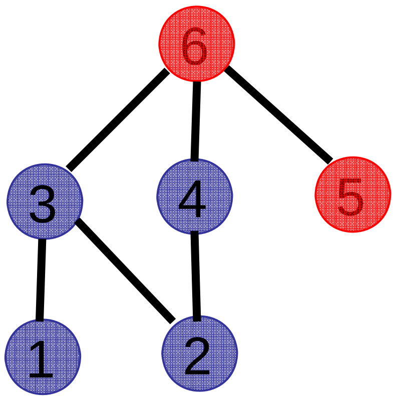
apply one control input at each leader $u = [u_1 \ldots u_m]^T$

Define $M = \mathbf{R}^{n\times m}$ where $M_{ij} = \begin{cases} 1 & \text{if } i = l_j \\ 0 & \text{otherwise} \end{cases}$

Overall dynamics

$$\dot{x}(t) = -Lx(t) + Mu$$

Example



$$\dot{x} = -\begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 2 & 0 & -1 & -1 & 0 & 0 \\ -1 & -1 & 3 & 0 & 0 & -1 \\ 0 & -1 & 0 & 2 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & -1 & -1 & 3 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} u$$

Leaders vs. followers

Choose *m* leaders: $\mathcal{V}_l = \{l_1, l_2, \ldots, l_m\} \subseteq \mathcal{V}$

apply one control input at each leader $u = [u_1 \ldots u_m]^T$

Define $M = \mathbf{R}^{n\times m}$ where $M_{ij} = \begin{cases} 1 & \text{if } i = l_j \\ 0 & \text{otherwise} \end{cases}$

Overall dynamics

$$\dot{x}(t) = -Lx(t) + Mu$$

Controllability $\Longleftrightarrow$ Graph topology

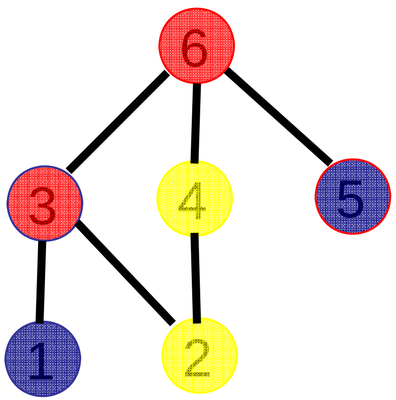
Graph partitions

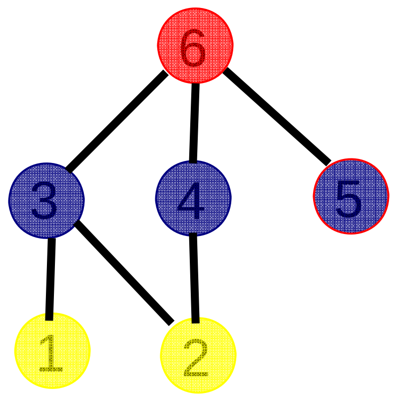
cell : collection of vertices $\mathcal{C}$

partition : collection of *mutually disjoint* cells $\pi = \{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_k\}$

discrete partition : a partition of *singleton* cells

equitable partition : a partition $\pi = \{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_k\}$ such that each node in $\mathcal{C}_i$ has the same number of neighbors in $\mathcal{C}_j$





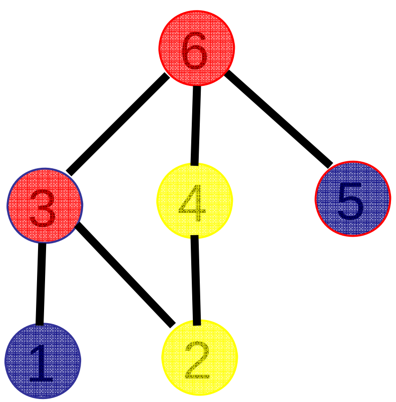
Graph partitions

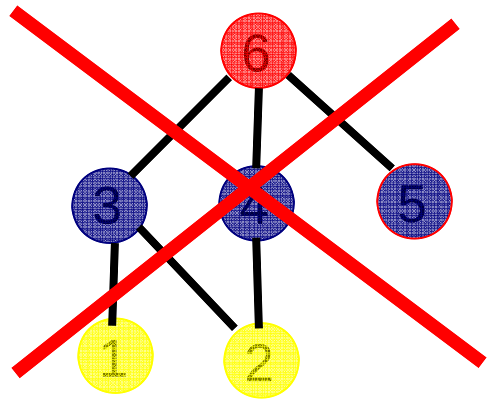
cell : collection of vertices $\mathcal{C}$

partition : collection of *mutually disjoint* cells $\pi = \{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_k\}$

discrete partition : a partition of *singleton* cells

equitable partition : a partition $\pi = \{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_k\}$ such that each node in $\mathcal{C}_i$ has the same number of neighbors in $\mathcal{C}_j$





Graph partitions

cell : collection of vertices $\mathcal{C}$

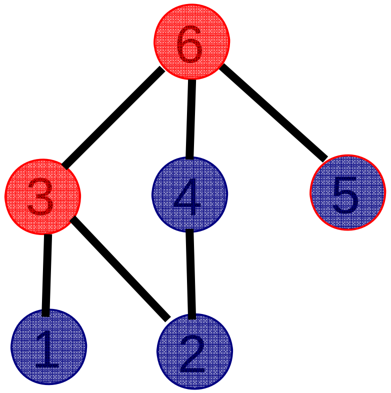
partition : collection of *mutually disjoint* cells $\pi = \{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_k\}$

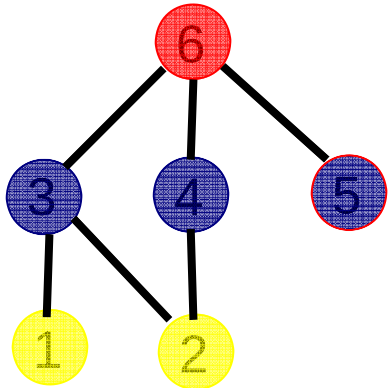
discrete partition : a partition of *singleton* cells

equitable partition : a partition $\pi = \{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_k\}$ such that each node in $\mathcal{C}_i$ has the same number of neighbors in $\mathcal{C}_j$

almost equitable partition : a partition $\pi = \{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_k\}$ such that each node in $\mathcal{C}_i$ has the same number of neighbors in $\mathcal{C}_j$

for all $i \neq j, \; i, j \in \{1, 2, \ldots, k\}$





Graph partitions

cell : collection of vertices $\mathcal{C}$

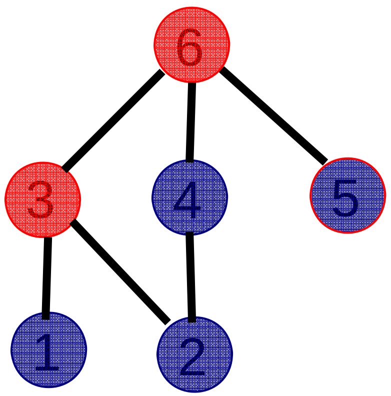
partition : collection of *mutually disjoint* cells $\pi = \{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_k\}$

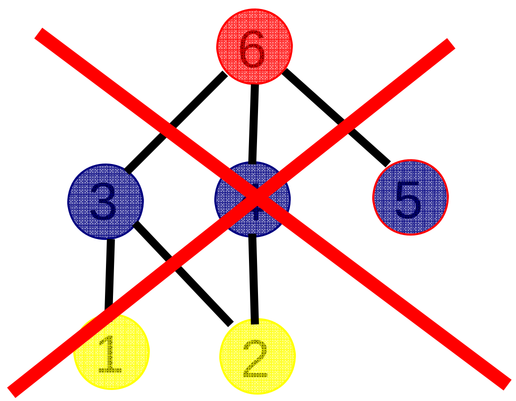
discrete partition : a partition of *singleton* cells

equitable partition : a partition $\pi = \{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_k\}$ such that each node in $\mathcal{C}_i$ has the same number of neighbors in $\mathcal{C}_j$

almost equitable partition : a partition $\pi = \{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_k\}$ such that each node in $\mathcal{C}_i$ has the same number of neighbors in $\mathcal{C}_j$

for all $i \neq j, \; i, j \in \{1, 2, \ldots, k\}$





Graph partitions

cell : collection of vertices $\mathcal{C}$

partition : collection of *mutually disjoint* cells $\pi = \{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_k\}$

discrete partition : a partition of *singleton* cells

equitable partition : a partition $\pi = \{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_k\}$ such that each node in $\mathcal{C}_i$ has the same number of neighbors in $\mathcal{C}_j$

almost equitable partition : a partition $\pi = \{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_k\}$ such that each node in $\mathcal{C}_i$ has the same number of neighbors in $\mathcal{C}_j$

for all $i \neq j,\ i, j \in \{1, 2, \ldots, k\}$

(*almost*) *equitable partition relative to v* : if it is a (almost) equitable partition and {*v*} is its cell

distance partition relative to v : if its cells consist of

$$\{u \in \mathcal{V} | \mathrm{dist}(u, v) = i\}$$

Pitfalls

Some elegant conclusions can be drawn using combinatorial characteristics

eg. necessary condition for controllability using graph symmetry

A. Rahmani, M. Ji, M. Mesbahi, and M. Egerstedt. Controllability of multi-agent systems from a graph theoretic perspective. *SIAM Journal on Control and Optimization*, 48:162-186, 2009

Sometimes the difficulty in such characterization is overlooked

eg. wrong sufficient conditions using almost equitable partitions

S. Martini, M. Egerstedt, and A. Bicchi. Controllability of multi-agent systems using relaxed equitable partitions. *International Journal of Systems, Control, and Communications*, 2:100-121, 2010

Lattice of partitions

Π : the set of all partitions for a given graph

$\pi_1 \leq \pi_2 \Longleftrightarrow$ every cell of π_1 is a subset of a cell of π_2

e.g. $\{\{1,2\},\{3\},\{4\},\{5\}\} \leq \{\{1,2,3\},\{4,5\}\}$

$\pi_1 \leq \pi_2 \Leftrightarrow \mathrm{card}(\pi_1) \geq \mathrm{card}(\pi_2)$

Π is a complete lattice, namely every subset of Π has an *infimum* and *supremum* within Π

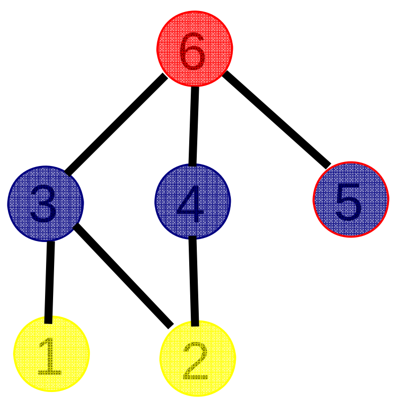
Notation: for $\Pi' \subseteq \Pi$, $\inf(\Pi')$ and $\sup(\Pi')$

Characteristic Matrix

Let $\pi = \{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_k\}$ be a partition.

Characteristic matrix $P(\pi)$ of π

$$p_{ij} = \begin{cases} 1 & \text{if } j \in \mathcal{C}_i \\ 0 & \text{otherwise} \end{cases}$$



$$\pi = \{\{1,2\},\{3,4,5\},\{6\}\}$$

$$P(\pi) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Facts

$$\begin{array}{ccc} \Pi_{EP} & \subseteq & \Pi_{AEP} \\ \cup\!\mid & & \cup\!\mid \\ \Pi_{EP}(v) & \subseteq & \Pi_{AEP}(v) \end{array}$$

$\pi \leq \pi_D(v)$ for all $v \in \mathcal{V}$ and for all $\pi \in \Pi_{AEP}(v)$

π is an EP $\Leftrightarrow \mathrm{im}P(\pi)$ is *A*-invariant

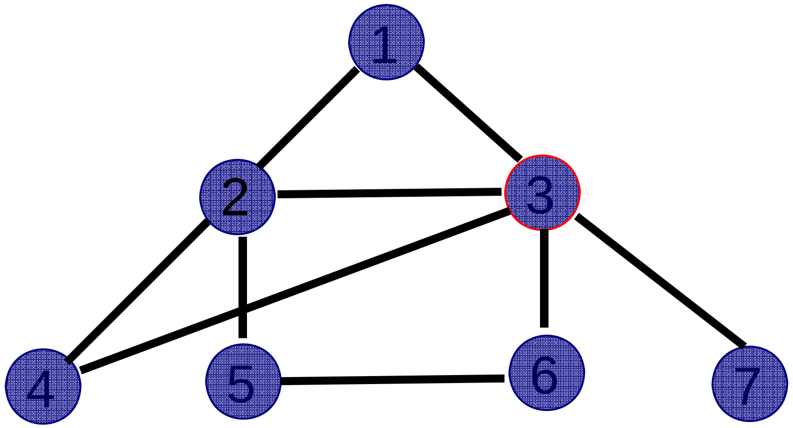
π is an AEP $\Leftrightarrow \mathrm{im}P(\pi)$ is *L*-invariant

$$\begin{aligned} \mathrm{im}P(\pi_1) \cap \mathrm{im}P(\pi_2) &= \mathrm{im}P(\sup\{\pi_1, \pi_2\}) \\ \mathrm{im}P(\pi_1) + \mathrm{im}P(\pi_2) &= \mathrm{im}P(\inf\{\pi_1, \pi_2\}) \end{aligned}$$

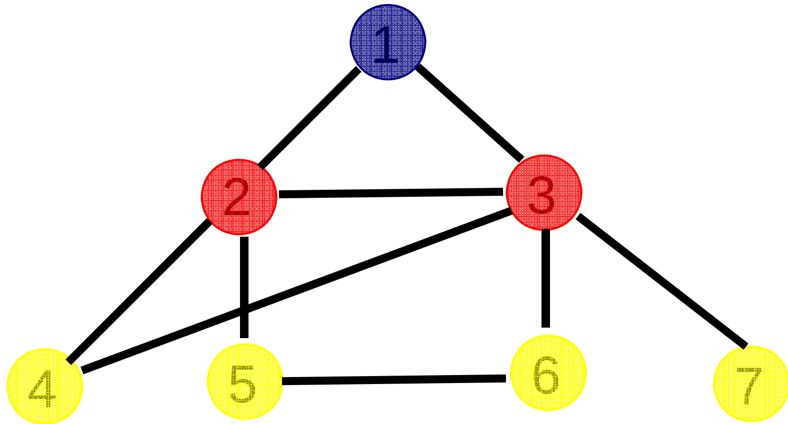
Use the following to denote the maximals of the corresponding partitions

$$\pi^*_{EP}, \pi^*_{EP}(v), \pi^*_{AEP}, \pi^*_{AEP(v)}$$

Examples

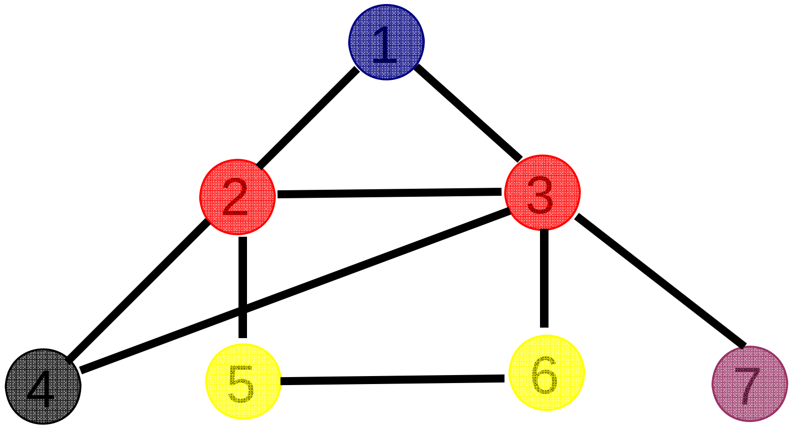


Examples



Distance partition $\pi_D(1)$

Examples



Equitable partition with respect to 1

Controllability Analysis

$$\dot{x} = -Lx + Mu$$

$\mathcal{K}$: the controllable subspace

Fact : $\mathcal{K}$ is the smallest L-invariant subspace that contains $\mathrm{im} M$

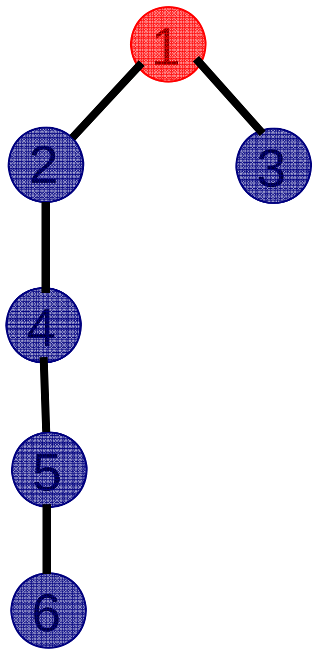
$$\mathcal{K} = \sum_{v \in \mathcal{V}_l} \mathcal{K}(v)$$

Single-leader case

$$\begin{aligned} \mathrm{card}(\pi_D(v)) &\leq \dim(\mathcal{K}(v)) \\ \mathcal{K}(v) &\subseteq \mathrm{im}P(\pi_{AEP}(v)) \end{aligned}$$

$$\mathrm{card}(\pi_D(v)) \leq \dim(\mathcal{K}(v)) \leq \dim\left(\mathrm{imP}(\pi^*_{AEP})(v)\right)$$

$5 = 5 < 6$

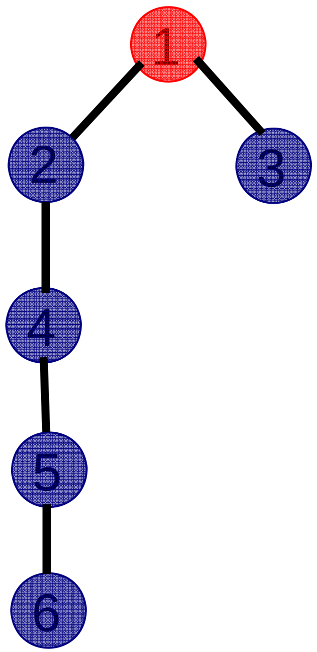


Single-leader case

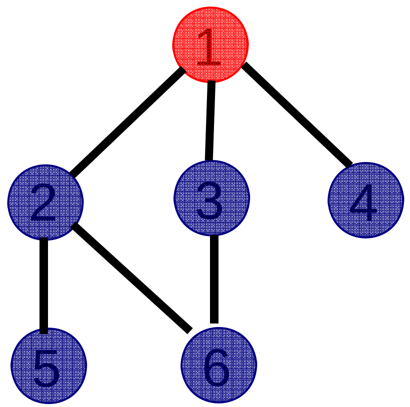
$$\begin{aligned} \mathrm{card}(\pi_D(v)) &\leq \dim(\mathcal{K}(v)) \\ \mathcal{K}(v) &= \mathrm{im}P(\pi^*_{AEP}(v)) \end{aligned}$$

$$\mathrm{card}(\pi_D(v)) \leq \dim(\mathcal{K}(v)) \leq \dim\left(\mathrm{imP}(\pi^*_{AEP})(v)\right)$$

$5 = 5 < 6$



$3 < 5 < 6$

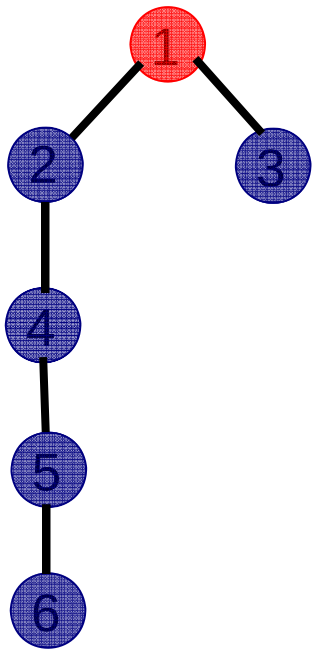


Single-leader case

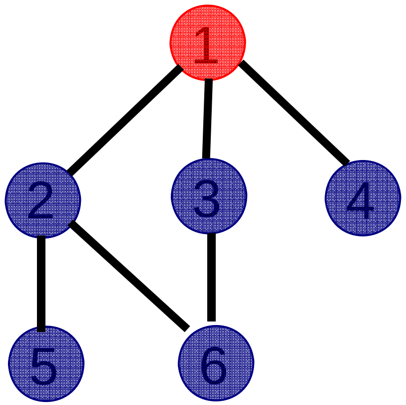
$$\begin{aligned} \mathrm{card}(\pi_D(v)) &\leq \dim(\mathcal{K}(v)) \\ \mathcal{K}(v) &= \mathrm{im}P(\pi^*_{AEP}(v)) \end{aligned}$$

$$\mathrm{card}(\pi_D(v)) \leq \dim(\mathcal{K}(v)) \leq \dim\left(\mathrm{imP}(\pi^*_{AEP})(v)\right)$$

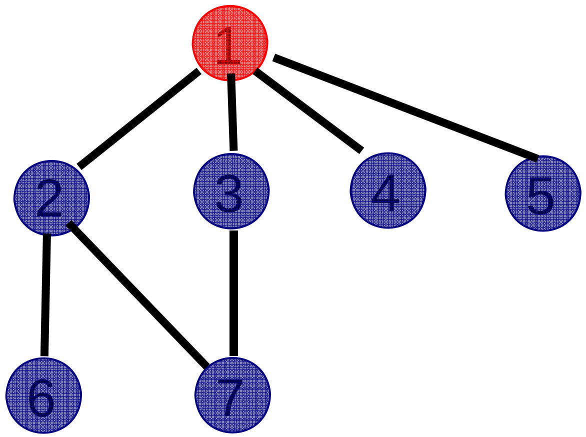
$5 = 5 < 6$



$3 < 5 < 6$



$3 < 6 = 6$

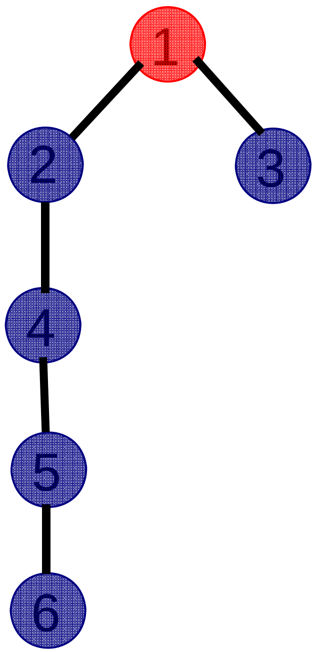


Single-leader case

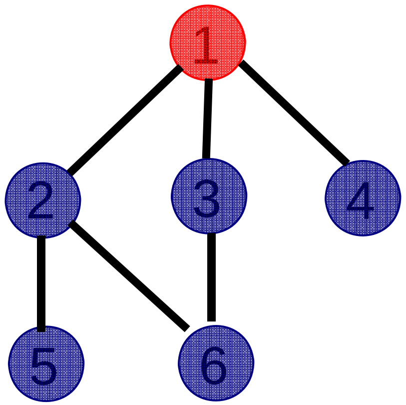
$$\begin{aligned} \mathrm{card}(\pi_D(v)) &\leq \dim(\mathcal{K}(v)) \\ \mathcal{K}(v) &= \mathrm{im}P(\pi^*_{AEP}(v)) \end{aligned}$$

$$\mathrm{card}(\pi_D(v)) \leq \dim(\mathcal{K}(v)) \leq \dim\left(\mathrm{imP}(\pi^*_{AEP})(v)\right)$$

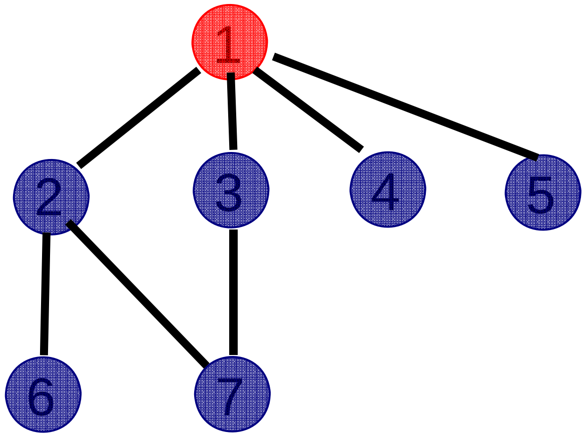
$5 = 5 < 6$



$3 < 5 < 6$



$3 < 6 = 6$



$\mathcal{K} \neq \mathrm{im}P(\pi)$ for any $\pi \in \Pi$!

Distance-regular graphs

$$\mathrm{diam}(\mathbf{G}) = \max\{\mathrm{dist}(u,v) | u,v \in \mathcal{V}\}$$

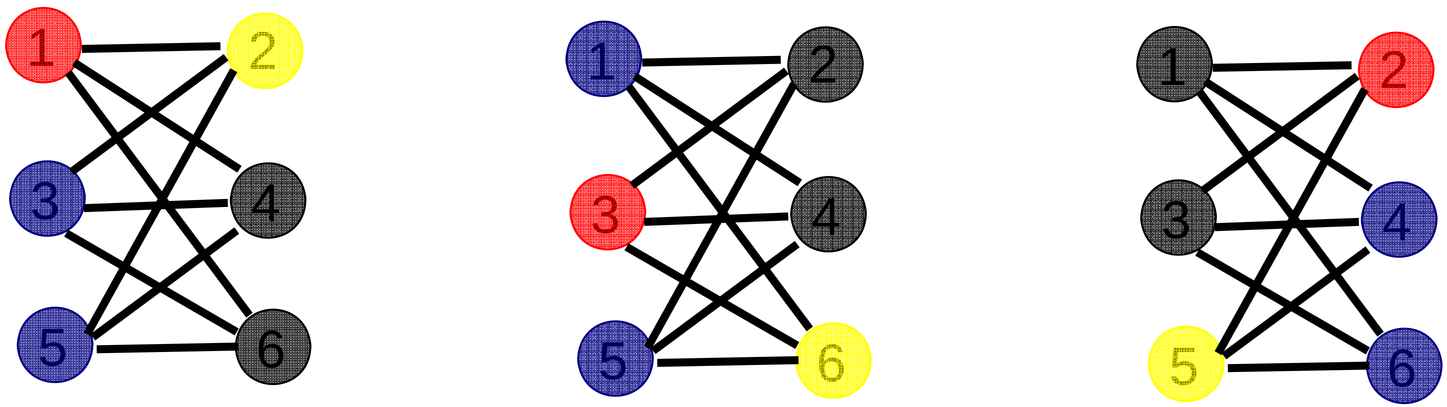
$\mathbf{G}$ is *distance-regular* if

for any vertices *u* and *v* and integers $i, j = 0, 1, \ldots, \mathrm{diam}(\mathbf{G})$,

the number of vertices at distance *i* from *u* and distance *j* from *v*

depends only on *i*, *j* and the distance between *u* and *v*,

independently of the choice of *u* and *v*.



$i = 1$ and $j = 2$

Under distance regularity

General graph

$$\Pi_{EP} \subseteq \Pi_{AEP}$$

$$\pi_D \geq \pi_{AEP}$$

$$\dim(\mathcal{K}(v)) \geq \mathrm{card}(\pi_D(v))$$

$$\mathcal{K}(v) \subseteq \mathrm{im} P(\pi^*_{AEP}(v))$$

Distance regular graph

$$\Pi_{EP} = \Pi_{AEP}$$

$$\pi_D(v) = \pi^*_{AEP}(v) \text{ for all } v \in \mathcal{V}$$

$$\dim((\mathcal{K}(v)) = \mathrm{diam}(\mathbf{G}) + 1$$

$$\mathcal{K}(v) = \mathrm{im} P(\pi^*_{AEP}(v))$$

If controllable, then $dm \geq n - 1$

Leader selection

1. Take $w \in \mathcal{V}$
2. Let $\pi_D(w) = \{\{w\}, \mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_d\}$
3. Take $w_i \in \mathcal{C}_i$ for $i = 1, 2, \ldots, d$
4. Let $\mathcal{V}_f = \{w_1, w_2, \ldots, w_d\}$
5. Define $\mathcal{V}_l = \mathcal{V} \setminus \mathcal{V}_f$

$$\boxed{\mathcal{K} = \sum_{v \in \mathcal{V}_l} \mathcal{K}(v) = \mathbf{R}^n}$$

Quick look at the proof

$$\begin{aligned}
\mathcal{K} &= \sum_{v \in \mathcal{V}_l} \mathcal{K}(v) \\
&= \sum_{v \in \mathcal{V}_l} \mathrm{im} P(\pi^*_{AEP}(v)) \\
&= \mathrm{im} P\left(\inf_{v \in \mathcal{V}_l} \pi^*_{AEP}(v) \right)
\end{aligned}$$

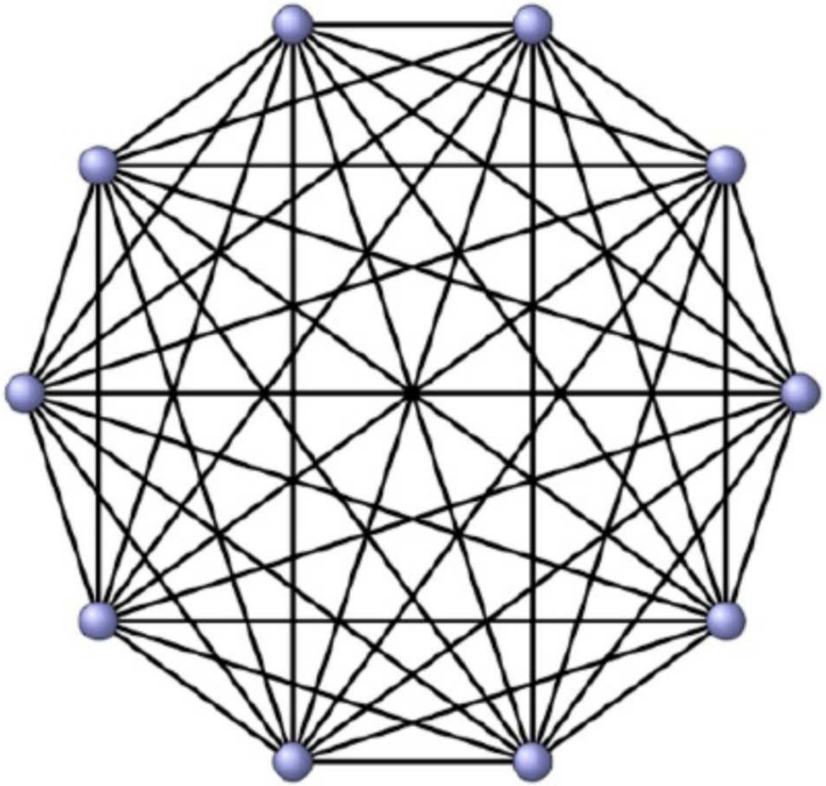
$\inf_{v \in \mathcal{V}_l} \pi^*_{AEP}(v)$ is discrete!

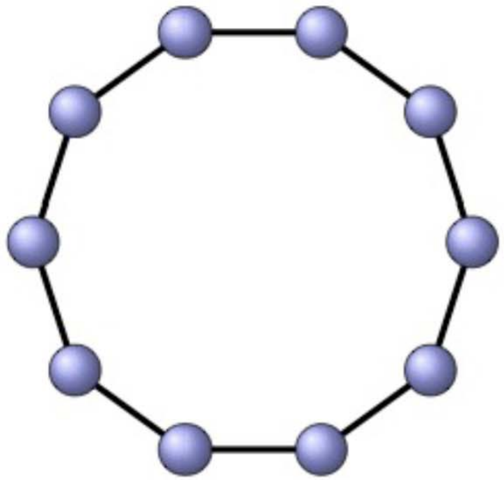
How many leaders needed?

1. Take $w \in \mathcal{V}$
2. Let $\pi_D(w) = \{\{w\}, \mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_d\}$
3. Take $w_i \mathcal{C}_i$ for $i = 1, 2, \ldots, d$
4. Let $\mathcal{V}_f = \{w_1, w_2, \ldots, w_d\}$
5. Define $\mathcal{V}_l = \mathcal{V} \setminus \mathcal{V}_f$

$$\boxed{\mathrm{card}\mathcal{V}_l = n - \mathrm{diam}(\mathbf{G})}$$

Complete and cycle graphs





$\text{diam} = 1, \text{ at least } n-1 \text{ leaders needed}$

$\text{diam} = \left\lfloor \dfrac{n}{2} \right\rfloor, \text{ at least 2 leaders needed}$

Concluding Remarks

- Controllability problem for multi-agent systems can be studied using graphs

- Tight bounds for controllability subspaces for systems with general graphs and distance regular graphs

- Future work
  - Better bounds on the least number of leaders for controllability
  - More complicated dynamics
  - Time-dependent graph topology

---- The End ----